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## HIGHER-DIMENSIONAL KINEMATICAL LIE ALGEBRAS VIA DEFORMATION THEORY

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ABSTRACT. We classify kinematical Lie algebras in dimension  $D \geq 4$ . This is approached via the classification of deformations of the relevant static kinematical Lie algebra. We also classify the deformations of the universal central extension of the static kinematical Lie algebra in dimension  $D \geq 4$ . In addition we determine which of these Lie algebras admit an invariant inner product.

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## 1. INTRODUCTION

In a previous paper [1] we have presented an approach to the classification of kinematical Lie algebras based on deformation theory, extending earlier work [2] for the galilean and Bargmann algebras. In [1] we recovered the classification of Bacry and Nuyts [3] of kinematical Lie algebras in dimension  $3+1$ , and extended it to classify also the deformations of the universal central extension of the static kinematical Lie algebra in that dimension. The purpose of this paper is to extend these classifications to dimension  $D+1$  for all  $D > 3$ . A separate paper [4] will present the classification of kinematical Lie algebras for  $D = 2$ , which is technically quite different than  $D = 3$  and  $D > 3$ . The results of this series of papers is summarised in [5].

By a **kinematical Lie algebra** in dimension  $D$ , we mean a real  $\frac{1}{2}(D+1)(D+2)$ -dimensional Lie algebra with generators  $R_{ab} = -R_{ba}$ , with  $1 \leq a, b \leq D$ , spanning a Lie subalgebra  $\mathfrak{s} \cong \mathfrak{so}(D)$ ; that is,

$$[R_{ab}, R_{cd}] = \delta_{bc}R_{ad} - \delta_{ac}R_{bd} - \delta_{bd}R_{ac} + \delta_{ad}R_{bc}, \quad (1)$$

and  $2D + 1$  generators  $B_a, P_a$  and  $H$  which transform according to the vector, vector and scalar representations of  $\mathfrak{so}(D)$ , respectively – namely,

$$\begin{aligned} [R_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b \\ [R_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \\ [R_{ab}, H] &= 0. \end{aligned} \quad (2)$$

The rest of the brackets between  $B_a, P_a$  and  $H$  are only subject to the Jacobi identity: in particular, they must be  $\mathfrak{s}$ -equivariant. The kinematical Lie algebra where those additional Lie brackets vanish is called the **static** kinematical Lie algebra. Every other kinematical Lie algebra will be, by definition, a deformation of the static one.

Up to isomorphism, there is only one kinematical Lie algebra in  $D = 0$ : it is one-dimensional and hence abelian. For  $D = 1$ , there are no rotations and hence any three-dimensional Lie algebra is kinematical. The classification is therefore the same as the celebrated Bianchi classification of three-dimensional real Lie algebras [6]. As far as I know the only other classification of kinematical Lie algebras is that in dimension  $D = 3$ , by Bacry and Nuyts [3]. The purpose of this paper is to solve the classification problem for  $D \geq 4$  using deformation theory along the lines of [2, 1]. As we will see, we can treat all  $D \geq 5$  in a uniform (with the exception for  $D = 5$  which is somewhat special but not in an essential way). The case  $D = 4$  is slightly more complicated due to  $\mathfrak{so}(4)$  being semisimple but not simple. Nevertheless as we will see, this case reduces to the generic ( $D \geq 5$ ) case; although this requires a calculation. The case  $D = 2$  is computationally more involved because  $\mathfrak{so}(2)$  is abelian and its vector representation (on  $\mathbb{R}^2$ ) has a larger than normal endomorphism ring. That case is the subject of a separate paper [4].

The static kinematical Lie algebra in  $D \geq 3$  admits a universal central extension with central generator  $Z$  and additional Lie bracket

$$[B_a, P_b] = \delta_{ab} Z, \quad (3)$$

and we will also consider the problem of classifying the deformations of the centrally extended static kinematical Lie algebra for  $D \geq 4$ , the case  $D = 3$  having been done in [1].

We refer to [1] for the methodology and the basic notions of deformation theory and Lie algebra cohomology as in [7] and [8] and [9].

This paper is organised as follows. In Section 2 we classify kinematical Lie algebras in dimension  $D + 1$  for all  $D \geq 5$ , arriving at Table 17. In Section 3 we treat the case of  $D = 4$ , but show after some calculations, that we obtain the same results as for  $D \geq 5$ . Therefore Table 17 is valid for  $D \geq 4$ . In Section 4 we determine the universal central extension of the static kinematical Lie algebra and proceed to classify its deformations for  $D \geq 5$ , arriving at Table 18. In Section 5 we repeat the calculation for  $D = 4$  arriving at the conclusion that Table 18 also holds for  $D \geq 4$ . Comparison with the case of  $D = 3$  shows that whereas in  $D = 3$  there are more kinematical Lie algebras which have no analogue in  $D > 3$ , the same is not true for the deformations of the centrally extended algebra. For those deformations, the results for  $D \geq 3$  are uniform. In Section 6 we offer some conclusions.

## 2. DEFORMATIONS OF THE STATIC KINEMATICAL LIE ALGEBRA WITH $D \geq 5$

Let  $\mathfrak{g}$  denote the static kinematical Lie algebra for  $D \geq 5$  and generators  $R_{ab}, B_a, P_a$  and  $H$ , subject to the nonzero brackets

$$\begin{aligned} [R_{ab}, R_{cd}] &= \delta_{bc} R_{ad} - \delta_{ac} R_{bd} - \delta_{bd} R_{ac} + \delta_{ad} R_{bc} \\ [R_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b \\ [R_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b. \end{aligned} \quad (4)$$

We often find it convenient to employ an abbreviated notation where the indices are suppressed. In that notation, we would write the above brackets as

$$[R, R] = R \quad [R, B] = B \quad \text{and} \quad [R, P] = P. \quad (5)$$

Let  $\mathfrak{h}$  denote the abelian ideal generated by  $B_a, P_a$  and  $H$ . Infinitesimal deformations of  $\mathfrak{g}$  are classified by the second Chevalley–Eilenberg cohomology group  $H^2(\mathfrak{g}; \mathfrak{g})$  which, by the Hochschild–Serre factorisation theorem, is isomorphic to  $H^2(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}}$ , where  $\mathfrak{s} \cong \mathfrak{so}(D)$  is the simple (since  $D \geq 5$ ) subalgebra generated by the  $R_{ab}$ . Since  $\mathfrak{s}$  is simple, we can compute this from the  $\mathfrak{s}$ -invariant complex

$$C^\bullet := C^\bullet(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}} = \{\mathfrak{s}\text{-equivariant linear maps } \wedge^\bullet \mathfrak{h} \rightarrow \mathfrak{g}\} \cong (\wedge^\bullet \mathfrak{h}^* \otimes \mathfrak{g})^{\mathfrak{s}}. \quad (6)$$

This shows that the Lie brackets involving the rotational generators do not deform and hence that every deformation of  $\mathfrak{g}$  is kinematical *a priori*. Let  $\beta_a, \pi_a$  and  $\eta$  denote the canonical dual basis for  $\mathfrak{h}^*$ . The differential  $\partial : C^p \rightarrow C^{p+1}$  is defined on generators by

$$\partial B_a = \partial P_a = \partial H = \partial \beta_a = \partial \pi_a = \partial \eta = 0 \quad \text{and} \quad \partial R_{ab} = \beta_a B_b - \beta_b B_a + \pi_a P_b - \pi_b P_a \quad (7)$$

and extended to  $C^\bullet$  as an odd derivation. It is clear by inspection that  $\partial^2 = 0$  on generators, and since it is an even derivation, it is identically 0.

We proceed to enumerate the cochains in  $C^p$  for  $p \leq 3$ . We will need to calculate the action of the differential  $\partial : C^1 \rightarrow C^2$  and  $\partial : C^2 \rightarrow C^3$  and in addition the Nijenhuis–Richardson brackets  $[-, -] : C^2 \times C^2 \rightarrow C^3$ .

**2.1. The Chevalley–Eilenberg cochains.** All  $\mathfrak{so}(D)$  invariant tensors are built out of  $\delta_{ab}$  and  $\epsilon_{a_1 \dots a_D}$ . We will use the Einstein summation convention in that repeated indices are summed over. In the absence of ambiguities, we will also use an abbreviated notation for cochains where we omit indices and assume that they are contracted with the invariant tensors in the only way possible.

The 0-cochains is the  $\mathfrak{s}$ -invariant subset of  $\mathfrak{g}$ , which is one-dimensional and spanned by  $H$ . Every 0-cochain is a cocycle, since  $\partial H = 0$ . There are no 1-coboundaries.

The 1-cochains are  $\mathfrak{s}$ -equivariant linear maps  $\mathfrak{h} \rightarrow \mathfrak{g}$ . A basis for the 1-cochains are given in Table 1, where we identify linear maps  $\mathfrak{h} \rightarrow \mathfrak{g}$  with elements of  $\mathfrak{h}^* \otimes \mathfrak{g}$  and where  $\eta H = \eta \otimes H$ ,  $\beta B = \beta_a \otimes B_a$ , et cetera. We see that all 1-cochains are cocycles and hence there are no 2-coboundaries.

TABLE 1. Basis for  $C^1(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}}$ 

$a_1$	$a_2$	$a_3$	$a_4$	$a_5$
$\eta H$	$\beta B$	$\beta P$	$\pi B$	$\pi P$

A basis for the 2-cochains are given in Table 2 as elements of  $\Lambda^2 \mathfrak{h}^* \otimes \mathfrak{g}$ , where  $\beta \pi R = \beta_a \wedge \pi_b \otimes R_{ab}$ , et cetera.

TABLE 2. Basis for  $C^2(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}}$ 

$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$
$\eta \beta B$	$\eta \beta P$	$\eta \pi B$	$\eta \pi P$	$\frac{1}{2} \beta \beta R$	$\beta \pi R$	$\frac{1}{2} \pi \pi R$	$\beta \pi H$

Finally, a basis for the 3-cochains are given in Table 3 as elements of  $\Lambda^3 \mathfrak{h}^* \otimes \mathfrak{g}$ , but in an abbreviated notation where, for example,  $\beta \pi \beta B = \beta_a \wedge \pi_b \wedge \beta_c \otimes B_b$ , et cetera. Those cochains in the second row are only present when  $D = 5$  and are given by  $\epsilon \beta \beta \beta R = \epsilon_{abcde} \beta_a \wedge \beta_b \wedge \beta_c \otimes R_{de}$ , et cetera. They will turn out to play no rôle in the calculations.

TABLE 3. Basis for  $C^3(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}}$ 

$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$	$b_7$	$b_8$
$\eta \beta \pi H$	$\eta \beta \beta R$	$\eta \beta \pi R$	$\eta \pi \pi R$	$\beta \pi \beta B$	$\beta \pi \beta P$	$\beta \pi \pi B$	$\beta \pi \pi P$
$b'_9$		$b'_{10}$	$b'_{11}$	$b'_{12}$			
$\epsilon \beta \beta \beta R$		$\epsilon \beta \beta \pi R$	$\epsilon \beta \pi \pi R$	$\epsilon \pi \pi \pi R$			

The last piece of data that we need is the Nijenhuis–Richardson bracket  $[-, -] : C^2 \times C^2 \rightarrow C^3$ , defined by

$$[[\lambda, \mu]] := \lambda \bullet \mu + \mu \bullet \lambda, \quad (8)$$

where  $\bullet$  is the operation defined on monomials by

$$(\alpha \otimes X) \bullet (\beta \otimes Y) := (\alpha \wedge \iota_X \beta) \otimes Y, \quad (9)$$

for  $\alpha, \beta \in \Lambda^2 \mathfrak{h}^*$  and  $X, Y \in \mathfrak{g}$ . Table 4 collects the calculations of  $c_i \bullet c_j$ , from where we can read off  $[[c_i, c_j]]$  by symmetrisation:

$$\begin{aligned} [[c_1, c_5]] &= b_2 & [[c_2, c_7]] &= b_3 & [[c_3, c_8]] &= b_7 \\ [[c_1, c_6]] &= b_3 & [[c_2, c_8]] &= b_6 & [[c_4, c_6]] &= b_3 \\ [[c_1, c_8]] &= b_1 + b_5 & [[c_3, c_5]] &= b_3 & [[c_4, c_7]] &= b_4 \\ [[c_2, c_6]] &= b_2 & [[c_3, c_6]] &= b_4 & [[c_4, c_8]] &= b_1 + b_8 \end{aligned} \quad (10)$$

**2.2. Infinitesimal deformations.** The action of the Chevalley–Eilenberg differential  $\partial : C^2 \rightarrow C^3$  is given by

$$\partial c_1 = \partial c_2 = \partial c_3 = \partial c_4 = \partial c_8 = 0 \quad \partial c_5 = -b_6 \quad \partial c_6 = b_5 - b_8 \quad \text{and} \quad \partial c_7 = b_7. \quad (11)$$

Therefore the five-dimensional space of 2-cocycles is spanned by  $c_1, c_2, c_3, c_4, c_8$ . Since there are no 2-coboundaries, this is also the cohomology. Therefore the most general infinitesimal deformation is given by a linear combination

$$\varphi_1 = t_1 c_1 + t_2 c_2 + t_3 c_3 + t_4 c_4 + t_5 c_8. \quad (12)$$

TABLE 4. Nijenhuis–Richardson •

•	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$
$c_1$	0	0	0	0	$b_2$	$b_3$	0	$b_1$
$c_2$	0	0	0	0	0	$b_2$	$b_3$	0
$c_3$	0	0	0	0	$b_3$	$b_4$	0	0
$c_4$	0	0	0	0	0	$b_3$	$b_4$	$b_1$
$c_5$	0	0	0	0	0	0	0	0
$c_6$	0	0	0	0	0	0	0	0
$c_7$	0	0	0	0	0	0	0	0
$c_8$	$b_5$	$b_6$	$b_7$	$b_8$	0	0	0	0

2.3. **Obstructions.** The first obstruction to integrability is given by

$$\frac{1}{2}[\varphi_1, \varphi_1] = t_1 t_5 (b_1 + b_5) + t_2 t_5 b_6 + t_3 t_5 b_7 + t_4 t_5 (b_1 + b_8). \quad (13)$$

Projecting to  $H^3$ , we see that  $[b_6] = [b_7] = 0$  and that  $[b_5] = [b_8]$ , so that

$$\left[\frac{1}{2}[\varphi_1, \varphi_1]\right] = (t_1 + t_4) t_5 ([b_1] + [b_5]), \quad (14)$$

so that the obstruction vanishes if and only if

$$t_5(t_1 + t_4) = 0. \quad (15)$$

If that equation is satisfied,  $\frac{1}{2}[\varphi_1, \varphi_1] = \partial\varphi_2$ , where

$$\varphi_2 = t_1 t_5 c_6 - t_2 t_5 c_5 + t_3 t_5 c_7. \quad (16)$$

The next obstruction is  $[\varphi_1, \varphi_2]$ , which is seen to vanish exactly provided that equation (15) is satisfied. This means that we can take  $\varphi_3 = 0$ . The next obstruction is  $\frac{1}{2}[\varphi_2, \varphi_2]$ , which is seen to vanish identically because  $c_5, c_6, c_7$  have vanishing Nijenhuis–Richardson brackets. This means that the deformation

$$\varphi = t_1 c_1 + t_2 c_2 + t_3 c_3 + t_4 c_4 - t_2 t_5 c_5 + t_1 t_5 c_6 + t_3 t_5 c_7 + t_5 c_8 \quad (17)$$

defines a Lie algebra provided that equation (15) is satisfied. That equation has two branches, depending on whether or not  $t_5 = 0$ .

2.4. **Deformations with  $t_5 = 0$ .** In this case, the deformation is

$$\varphi = t_1 c_1 + t_2 c_2 + t_3 c_3 + t_4 c_4, \quad (18)$$

which leads to the Lie brackets (in abbreviated notation):

$$\begin{aligned} [H, B] &= t_1 B + t_2 P \\ [H, P] &= t_3 B + t_4 P. \end{aligned} \quad (19)$$

As in the  $D = 3$  case, we can bring these to normal forms depending on the value of the discriminant  $\delta := (t_1 - t_4)^2 + 4t_2 t_3$ :

(1)  $\delta > 0$  (or  $\delta = 0$  and diagonalisable):

$$\boxed{[H, B] = \gamma B \quad \text{and} \quad [H, P] = P,} \quad (20)$$

where  $\gamma \in [-1, 1]$ . The case  $\gamma = -1$  is the higher-dimensional version of the **(lorentzian) Newton** Lie algebra.

(2)  $\delta = 0$  (and not diagonalisable):

$$\boxed{[H, B] = B + P \quad \text{and} \quad [H, P] = P,} \quad (21)$$

or

$$\boxed{[H, B] = P,} \quad (22)$$

which is the **galilean** algebra.

(3)  $\delta < 0$ :

$$\boxed{[H, B] = \alpha B + P \quad \text{and} \quad [H, P] = -B + \alpha P,} \quad (23)$$

where  $\alpha \geq 0$ . The case  $\alpha = 0$  is the higher-dimensional version of the **(euclidean) Newton** Lie algebra.

**2.5. Deformations with  $t_5 \neq 0$ .** In this case, equation (15) forces  $t_1 + t_4 = 0$ , so that the deformation is

$$\varphi = t_1(c_1 - c_4) + t_2c_2 + t_3c_3 - t_2t_5c_5 + t_1t_5c_6 + t_3t_5c_7 + t_5c_8, \quad (24)$$

which leads to the Lie brackets

$$\begin{aligned} [H, B] &= t_1B + t_2P & [B, B] &= -t_2t_5R & [B, P] &= t_5H + t_1t_5R. \\ [H, P] &= t_3B - t_1P & [P, P] &= t_3t_5R \end{aligned} \quad (25)$$

In order to bring these Lie brackets to normal form, it will prove useful to study the action of those automorphisms of  $\mathfrak{h}$  which commute with the action of  $\mathfrak{s}$ . This is similar to what happens in  $D = 3$  and we refer to [1] for a more detailed description of the method.

The subgroup  $G$  of automorphisms of  $\mathfrak{h}$  which commutes with the  $\mathfrak{s}$ -action is  $GL(\mathbb{R}^2) \times \mathbb{R}^\times$  acting on generators as follows:

$$(\mathbf{B}, \mathbf{P}, H) \mapsto (\mathbf{B}, \mathbf{P}, H) \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(\mathbb{R}^2) \quad \text{and} \quad \lambda \in \mathbb{R}^\times. \quad (26)$$

The induced action on  $\mathfrak{h}^*$  is given by

$$(\beta, \pi, \eta) \mapsto (\beta, \pi, \eta) \begin{pmatrix} d/\Delta & -c/\Delta & 0 \\ -b/\Delta & a/\Delta & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \quad \text{where} \quad \Delta = ad - bc. \quad (27)$$

From this one reads off how  $G$  acts on  $C^2$ :

$$\begin{aligned} c_1 + c_4 &\mapsto \frac{1}{\lambda}(c_1 + c_4) \\ c_1 - c_4 &\mapsto \frac{1}{\lambda\Delta}((ad + bc)(c_1 - c_4) + 2cdc_2 - 2abc_3) \\ c_2 &\mapsto \frac{1}{\lambda\Delta}(bd(c_1 - c_4) + d^2c_2 - b^2c_3) \\ c_3 &\mapsto \frac{1}{\lambda\Delta}(-ac(c_1 - c_4) - c^2c_2 + a^2c_3) \\ c_5 &\mapsto \frac{1}{\Delta^2}(d^2c_5 - bdc_6 + b^2c_7) \\ c_6 &\mapsto \frac{1}{\Delta^2}(-2cdc_5 + (ad + bc)c_6 - 2abc_7) \\ c_7 &\mapsto \frac{1}{\Delta^2}(c^2c_5 - acc_6 + a^2c_7) \\ c_8 &\mapsto \lambda\Delta c_8, \end{aligned} \quad (28)$$

and from this we arrive at how  $G$  acts on the deformation parameters  $(t_1, t_2, t_3, t_5)$ :

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_5 \end{pmatrix} \mapsto \frac{1}{\lambda\Delta} \begin{pmatrix} ad + bc & bd & -ac & 0 \\ 2cd & d^2 & -c^2 & 0 \\ -2ab & -b^2 & a^2 & 0 \\ 0 & 0 & 0 & \lambda^2\Delta^2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_5 \end{pmatrix}. \quad (29)$$

The representation  $\rho$  of  $GL(\mathbb{R}^2)$  defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{\Delta} \begin{pmatrix} ad + bc & bd & -ac \\ 2cd & d^2 & -c^2 \\ -2ab & -b^2 & a^2 \end{pmatrix} \quad (30)$$

is not faithful – having as kernel the scalar matrices – and preserves the lorentzian inner product

$$K = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (31)$$

With a suitable choice of  $\lambda$ , we can use such three-dimensional Lorentz transformations to bring  $\mathbf{t} = (t_1, t_2, t_3)$  to one of the following normal forms, each one labelling an orbit of  $G$  on the space of such parameters:

- (1) the **zero** orbit, where  $\mathbf{t} = (0, 0, 0)$ ;
- (2) the **spacelike** orbit, where  $\mathbf{t} = (1, 0, 0)$ ;
- (3) the **timelike** orbit, where  $\mathbf{t} = (0, 1, -1)$ ; and
- (4) the **lightlike** orbit, where  $\mathbf{t} = (0, 0, 1)$ .

This still leaves the possibility to act with a (nonzero) scalar matrix to set  $t_5 = \pm 1$ , since scalar matrices in  $GL(\mathbb{R}^2)$  have positive determinant. We discuss each one of these cases in turn.

**2.5.1. The zero branch.** In this case we can actually set  $t_5 = 1$  without loss of generality. The nonzero Lie brackets are

$$\boxed{[B, P] = H}, \quad (32)$$

which is the higher-dimensional analogue of the **Carroll** algebra.

2.5.2. *The spacelike branch.* Here we can also set  $t_5 = 1$  without loss of generality, and the nonzero Lie brackets are

$$[H, B] = B \quad [H, P] = -P \quad \text{and} \quad [B, P] = H + R. \quad (33)$$

This Lie algebra is isomorphic to  $\mathfrak{so}(D+1, 1)$ . If we change basis so that  $(B, P) \mapsto (\frac{1}{\sqrt{2}}(B+P), \frac{1}{\sqrt{2}}(B-P))$ , then it takes the more standard form

$$\boxed{[H, B] = P \quad [H, P] = B \quad [B, B] = R \quad [B, P] = -H \quad \text{and} \quad [P, P] = -R.} \quad (34)$$

2.5.3. *The timelike branch.* In this case, the nonzero Lie brackets are, for  $\varepsilon = \pm 1$ ,

$$\boxed{\begin{array}{lll} [H, B] = P & [B, B] = \varepsilon R & \\ [H, P] = -B & [P, P] = \varepsilon R & [B, P] = -\varepsilon H. \end{array}} \quad (35)$$

These Lie algebras are isomorphic to  $\mathfrak{so}(D+2)$  (for  $\varepsilon = -1$ ) or  $\mathfrak{so}(D, 2)$  (for  $\varepsilon = 1$ ).

2.5.4. *The lightlike branch.* In this case, the nonzero Lie brackets are, for  $\varepsilon = \pm 1$ ,

$$\boxed{[H, P] = \varepsilon B \quad [P, P] = \varepsilon R \quad \text{and} \quad [B, P] = H,} \quad (36)$$

after redefining  $H$ . These Lie algebras are isomorphic to the euclidean Lie algebra  $\mathfrak{e}$  for  $\varepsilon = -1$ , or the Poincaré Lie algebra  $\mathfrak{p}$  for  $\varepsilon = 1$ .

2.6. **Invariant inner products.** We shall now investigate the existence of invariant inner products on the kinematical Lie algebras determined in this section. We remind the reader that by an invariant inner product on a Lie algebra  $\mathfrak{g}$  we mean a nondegenerate symmetric bilinear form  $(-, -) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  which is “associative”; that is,

$$([x, y], z) = (x, [y, z]) \quad \forall x, y, z \in \mathfrak{g}. \quad (37)$$

The Killing form is associative, but by Cartan’s semisimplicity criterion, it is only nondegenerate for semisimple Lie algebras. This means that the simple Lie algebras  $\mathfrak{so}(D+1, 1)$ ,  $\mathfrak{so}(D+2)$  and  $\mathfrak{so}(D, 2)$  admit invariant inner products: namely, any nonzero multiple of the Killing form. It will turn out that these are the only kinematical Lie algebras which do. To prove this, rather than appealing to any general structural results, we will simply exploit the associativity condition (37).

We shall first of all show that no kinematical Lie algebra where  $B$  and  $P$  span an abelian ideal can admit an invariant inner product. This will rule out the first five rows in Table 17. Indeed, let  $(-, -)$  be an associative symmetric bilinear form. We will show that it is degenerate. To this end, let  $X, Y$  be any of  $B, P$  and consider (in abbreviated notation)

$$(X, Y) = ([R, X], Y) = (R, [X, Y]) = 0, \quad (38)$$

where we have used associativity and the fact that  $X, Y$  are vectors under rotations. By rotational invariance, the only possible nonzero inner products involving  $B$  and  $P$  are of the form  $(B, B)$ ,  $(B, P)$  and  $(P, P)$ , and as we have just seen, these are zero. Therefore  $(-, -)$  is degenerate.

Any associative symmetric bilinear form in the Carroll algebra is degenerate, since  $(H, -) = 0$ . Indeed, by rotational invariance, the only possible nonzero inner product of  $H$  is with itself, but then

$$\delta_{ab}(H, H) = (H, [B_a, P_b]) = ([H, B_a], P_b) = 0. \quad (39)$$

It remains to consider the euclidean and Poincaré algebras. So let  $(-, -)$  be an associative symmetric bilinear form on either  $\mathfrak{e}$  or  $\mathfrak{p}$  and let us calculate  $(H, H)$ , which is the only possibly nonzero rotationally invariant inner product involving  $H$ :

$$\delta_{ab}(H, H) = ([B_a, P_b], H) = (B_a, [P_b, H]) = 0. \quad (40)$$

### 3. KINEMATICAL LIE ALGEBRAS WITH $D = 4$

In this section we will let  $\mathfrak{g}$  denote the static kinematical Lie algebra for  $D = 4$ . The case  $D = 4$  is slightly more complicated due to the fact that the rotation subalgebra  $\mathfrak{so}(4)$  is not simple, but rather  $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ . This is due to the existence of the Hodge star, an  $\mathfrak{so}(4)$ -invariant linear map  $\star : \Lambda^2 \mathbb{R}^4 \rightarrow \Lambda^2 \mathbb{R}^4$  which obeys  $\star^2 = \mathbb{1}$  and hence decomposes  $\Lambda^2 \mathbb{R}^4$  into its eigenspaces  $\Lambda_{\pm}^2$ , each one corresponding to an  $\mathfrak{so}(3)$  subalgebra.

Let  $R_{ab}^{\pm} := \frac{1}{2}(R_{ab} \pm \frac{1}{2}\epsilon_{abcd}R_{cd})$  span  $\mathfrak{s} \cong \mathfrak{so}(4)$  and  $B_a, P_a, H$  span the abelian ideal  $\mathfrak{h}$  of  $\mathfrak{g}$ . As usual we will choose the canonical dual basis  $\beta_a, \pi_a, \eta$  for  $\mathfrak{h}^*$ .

The nonzero Lie brackets in that basis are given by

$$\begin{aligned} [R_{ab}^{\pm}, R_{cd}^{\pm}] &= [R_{ab}, R_{cd}]^{\pm} \\ [R_{ab}^{\pm}, B_c] &= \frac{1}{2}(\delta_{bc}B_a - \delta_{ac}B_b \mp \epsilon_{abcd}B_d) \\ [R_{ab}^{\pm}, P_c] &= \frac{1}{2}(\delta_{bc}P_a - \delta_{ac}P_b \mp \epsilon_{abcd}P_d) \end{aligned} \quad (41)$$



**3.1. The Chevalley–Eilenberg complex.** The fact that  $\mathfrak{s}$  is semisimple suffices for the Hochschild–Serre decomposition theorem and we may calculate the infinitesimal deformations from the  $\mathfrak{s}$ -invariant subcomplex  $C^\bullet := C^\bullet(\mathfrak{h}; \mathfrak{g})^\mathfrak{s}$ . In particular this shows that all deformations are automatically kinematical.

We now proceed to enumerate bases for the spaces of cochains, noting that  $C^0$  is spanned by  $H$ . The dimensions of  $C^1$ ,  $C^2$  and  $C^3$  are 5, 11 and 19, respectively. Natural bases are tabulated below. In Table 7, the cochains in the second row involve the  $\epsilon$  tensor, so that, for example,  $\epsilon\beta\beta\pi B = \epsilon_{abcd}\beta_a \wedge \beta_b \wedge \pi_c \otimes B_d$ , et cetera.

TABLE 5. Basis for  $C^1(\mathfrak{h}; \mathfrak{g})^\mathfrak{s}$ 

$a_1$	$a_2$	$a_3$	$a_4$	$a_5$
$\eta H$	$\beta B$	$\beta P$	$\pi B$	$\pi P$

TABLE 6. Basis for  $C^2(\mathfrak{h}; \mathfrak{g})^\mathfrak{s}$ 

$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$	$c_9$	$c_{10}$	$c_{11}$
$\eta\beta B$	$\eta\beta P$	$\eta\pi B$	$\eta\pi P$	$\frac{1}{2}\beta\beta R^+$	$\frac{1}{2}\beta\beta R^-$	$\beta\pi R^+$	$\beta\pi R^-$	$\frac{1}{2}\pi\pi R^+$	$\frac{1}{2}\pi\pi R^-$	$\beta\pi H$

TABLE 7. Basis for  $C^3(\mathfrak{h}; \mathfrak{g})^\mathfrak{s}$ 

$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$	$b_7$	$b_8$	$b_9$	$b_{10}$	$b_{11}$
$\eta\beta\pi H$	$\eta\beta\beta R^+$	$\eta\beta\beta R^-$	$\eta\beta\pi R^+$	$\eta\beta\pi R^-$	$\eta\pi\pi R^+$	$\eta\pi\pi R^-$	$\beta\pi\beta B$	$\beta\pi\beta P$	$\beta\pi\pi B$	$\beta\pi\pi P$
$b_{12}$	$b_{13}$	$b_{14}$	$b_{15}$	$b_{16}$	$b_{17}$	$b_{18}$	$b_{19}$			
$\epsilon\beta\beta\beta B$	$\epsilon\beta\beta\beta P$	$\epsilon\beta\beta\pi B$	$\epsilon\beta\beta\pi P$	$\epsilon\beta\pi\pi B$	$\epsilon\beta\pi\pi P$	$\epsilon\pi\pi\pi B$	$\epsilon\pi\pi\pi P$			

The Chevalley–Eilenberg differential  $\partial : C^p \rightarrow C^{p+1}$  is defined on generators by

$$\partial\eta = \partial\beta_a = \partial\pi_a = \partial H = \partial B_a = \partial P_a = 0 \quad (42)$$

and

$$\partial R_{ab}^\pm = \frac{1}{2} (\beta_a B_b - \beta_b B_a + \pi_a P_b - \pi_b P_a \pm \epsilon_{abcd} (\beta_c B_d + \pi_c P_d)). \quad (43)$$

In particular, it follows that  $\partial$  is identically zero on  $C^0$  and  $C^1$ , so that  $B^2 = 0$ . On  $C^2$  we find

$$\begin{aligned} \partial c_5 &= -\frac{1}{2}b_9 + \frac{1}{4}(b_{12} + b_{15}) & \partial c_7 &= \frac{1}{2}(b_8 - b_{11} + b_{14} + b_{17}) & \partial c_9 &= \frac{1}{2}b_{10} + \frac{1}{4}(b_{16} + b_{19}) \\ \partial c_6 &= -\frac{1}{2}b_9 - \frac{1}{4}(b_{12} + b_{15}) & \partial c_8 &= \frac{1}{2}(b_8 - b_{11} - b_{14} - b_{17}) & \partial c_{10} &= \frac{1}{2}b_{10} - \frac{1}{4}(b_{16} + b_{19}). \end{aligned} \quad (44)$$

The last piece of data that we need is the Nijenhuis–Richardson bracket  $\llbracket -, - \rrbracket : C^2 \times C^2 \rightarrow C^3$ . Table 8 collects the calculations of  $c_i \bullet c_j$ , from where we can read off  $\llbracket c_i, c_j \rrbracket$  by symmetrisation:

$$\begin{aligned} \llbracket c_1, c_5 \rrbracket &= b_2 & \llbracket c_2, c_7 \rrbracket &= b_2 & \llbracket c_3, c_5 \rrbracket &= b_4 & \llbracket c_4, c_7 \rrbracket &= b_4 \\ \llbracket c_1, c_6 \rrbracket &= b_3 & \llbracket c_2, c_8 \rrbracket &= b_3 & \llbracket c_3, c_6 \rrbracket &= b_5 & \llbracket c_4, c_8 \rrbracket &= b_5 \\ \llbracket c_1, c_7 \rrbracket &= b_4 & \llbracket c_2, c_9 \rrbracket &= b_4 & \llbracket c_3, c_7 \rrbracket &= b_6 & \llbracket c_4, c_9 \rrbracket &= b_6 \\ \llbracket c_1, c_8 \rrbracket &= b_5 & \llbracket c_2, c_{10} \rrbracket &= b_5 & \llbracket c_3, c_8 \rrbracket &= b_7 & \llbracket c_4, c_{10} \rrbracket &= b_7 \\ \llbracket c_1, c_{11} \rrbracket &= b_1 + b_8 & \llbracket c_2, c_{11} \rrbracket &= b_9 & \llbracket c_3, c_{11} \rrbracket &= b_{10} & \llbracket c_4, c_{11} \rrbracket &= b_1 + b_{11}. \end{aligned} \quad (45)$$

**3.2. Infinitesimal deformations.** From the action of the Chevalley–Eilenberg differential  $\partial$  on  $C^2$ , described in equation (44), we find that

$$H^2 \cong Z^2 = \mathbb{R} \langle c_1, c_2, c_3, c_4, c_{11} \rangle. \quad (46)$$

Therefore the most general infinitesimal deformation is

$$\varphi_1 = t_1 c_1 + t_2 c_2 + t_3 c_3 + t_4 c_4 + t_5 c_{11}. \quad (47)$$



TABLE 8. Nijenhuis–Richardson •

•	c <sub>1</sub>	c <sub>2</sub>	c <sub>3</sub>	c <sub>4</sub>	c <sub>5</sub>	c <sub>6</sub>	c <sub>7</sub>	c <sub>8</sub>	c <sub>9</sub>	c <sub>10</sub>	c <sub>11</sub>
c <sub>1</sub>	0	0	0	0	b <sub>2</sub>	b <sub>3</sub>	b <sub>4</sub>	b <sub>5</sub>	0	0	b <sub>1</sub>
c <sub>2</sub>	0	0	0	0	0	0	b <sub>2</sub>	b <sub>3</sub>	b <sub>4</sub>	b <sub>5</sub>	0
c <sub>3</sub>	0	0	0	0	b <sub>4</sub>	b <sub>5</sub>	b <sub>6</sub>	b <sub>7</sub>	0	0	0
c <sub>4</sub>	0	0	0	0	0	0	b <sub>4</sub>	b <sub>5</sub>	b <sub>6</sub>	b <sub>7</sub>	b <sub>1</sub>
c <sub>5</sub>	0	0	0	0	0	0	0	0	0	0	0
c <sub>6</sub>	0	0	0	0	0	0	0	0	0	0	0
c <sub>7</sub>	0	0	0	0	0	0	0	0	0	0	0
c <sub>8</sub>	0	0	0	0	0	0	0	0	0	0	0
c <sub>9</sub>	0	0	0	0	0	0	0	0	0	0	0
c <sub>10</sub>	0	0	0	0	0	0	0	0	0	0	0
c <sub>11</sub>	b <sub>8</sub>	b <sub>9</sub>	b <sub>10</sub>	b <sub>11</sub>	0	0	0	0	0	0	0

3.3. **Obstructions.** The first obstruction is the class in  $H^3$  of  $\frac{1}{2}[\varphi_1, \varphi_1]$ , which we can calculate from the explicit expressions (45) for the Nijenhuis–Richardson bracket. Doing so, we find

$$\frac{1}{2}[\varphi_1, \varphi_1] = t_1 t_5 (b_1 + b_8) + t_2 t_5 b_9 + t_3 t_5 b_{10} + t_4 t_5 (b_1 + b_{11}). \quad (48)$$

From equation (44) we learn that

$$B^3 = \mathbb{R} \langle b_8 - b_{11}, b_9, b_{10}, b_{12} + b_{15}, b_{14} + b_{17}, b_{16} + b_{19} \rangle, \quad (49)$$

so that in cohomology,  $[b_8] = [b_{11}]$ ,  $[b_9] = [b_{10}] = 0$ ,  $[b_{12}] = -[b_{15}]$ ,  $[b_{14}] = -[b_{17}]$  and  $[b_{16}] = -[b_{19}]$ . Therefore in  $H^3$ ,

$$[\frac{1}{2}[\varphi_1, \varphi_1]] = (t_1 + t_4) t_5 ([b_1] + [b_8]), \quad (50)$$

so that the obstruction vanishes if and only if

$$(t_1 + t_4) t_5 = 0. \quad (51)$$

If this equation is satisfied,  $\frac{1}{2}[\varphi_1, \varphi_1] = \partial \varphi_2$  for

$$\varphi_2 = t_1 t_5 (c_7 + c_8) - t_2 t_5 (c_5 + c_6) + t_3 t_5 (c_9 + c_{10}). \quad (52)$$

The next obstruction  $[\varphi_1, \varphi_2]$  vanishes identically using (51), so we can take  $\varphi_3 = 0$ . The next obstruction is  $\frac{1}{2}[\varphi_2, \varphi_2]$ , but this also vanishes identically, so that  $\varphi_4 = 0$  and hence there are no further obstructions. In summary, the most general deformation is given by

$$\varphi = t_1 c_1 + t_2 c_2 + t_3 c_3 + t_4 c_4 + t_5 c_{11} + t_1 t_5 (c_7 + c_8) - t_2 t_5 (c_5 + c_6) + t_3 t_5 (c_9 + c_{10}), \quad (53)$$

subject to equation (51).

Notice that since  $R^+ + R^- = R$ , the sums of cochains appearing in  $\varphi$  are

$$c_5 + c_6 = \frac{1}{2} \beta \beta R \quad c_7 + c_8 = \beta \pi R \quad \text{and} \quad c_9 + c_{10} = \frac{1}{2} \pi \pi R, \quad (54)$$

which means that the expression (53) above for  $\varphi$  coincides *mutatis mutandis* with the one in equation (17). Since the conditions (51) and (15) are identical, the rest of the analysis proceeds as in the case  $D \geq 5$ . As in  $D \geq 5$ , only the simple kinematical Lie algebras admit an associative inner product.

#### 4. DEFORMATIONS OF THE CENTRALLY EXTENDED STATIC KINEMATICAL LIE ALGEBRA WITH $D \geq 5$

Let  $D \geq 5$  and let us consider the static kinematical Lie algebra  $\mathfrak{g}$  defined (in abbreviated form) by equation (5). We shall show that it has a central extension with bracket

$$[B_a, P_b] = \delta_{ab} Z \quad (\text{or in abbreviated form } [B, P] = Z.) \quad (55)$$

We will then classify the deformations of the centrally extended Lie algebra  $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}Z$ .

**4.1. Central extension of the static kinematical Lie algebra.** Central extensions of  $\mathfrak{g}$  are classified by  $H^2(\mathfrak{g}; \mathbb{R})$ , which by the Hochschild–Serre factorisation theorem is isomorphic to  $H^2(\mathfrak{h}; \mathbb{R})^{\mathfrak{s}}$  and this in turn can be computed from the subcomplex  $C^\bullet$  of  $\mathfrak{s}$ -invariant cochains in  $\Lambda^\bullet \mathfrak{h}^*$ . The first three spaces of cochains in the subcomplex are

$$C^1 = \mathbb{R} \langle \eta \rangle \quad C^2 = \mathbb{R} \langle \beta\pi \rangle \quad \text{and} \quad C^3 = \mathbb{R} \langle \eta\beta\pi \rangle, \quad (56)$$

where we again use the abbreviated notation  $\beta\pi = \beta_a \wedge \pi_a$ , et cetera. Since  $\mathfrak{h}$  is abelian, the differential is identically zero, so that  $H^2 = C^2$  with cocycle representative  $\beta\pi$ . The universal central extension  $\tilde{\mathfrak{g}}$  of  $\mathfrak{g}$  is thus spanned by  $R_{ab}, B_a, P_a, H, Z$  with nonzero brackets

$$[R, R] = R \quad [R, B] = B \quad [R, P] = P \quad \text{and} \quad [B, P] = Z. \quad (57)$$

**4.2. The deformation complex.** Let  $\tilde{\mathfrak{h}}$  denote the ideal of  $\tilde{\mathfrak{g}}$  spanned by  $B_a, P_a, H, Z$  and let  $\mathfrak{s} \cong \mathfrak{so}(D)$  again be the rotational subalgebra. By Hochschild–Serre, the deformation complex can be taken to be the  $\mathfrak{s}$ -invariant subcomplex  $C^\bullet = C^\bullet(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^{\mathfrak{s}}$ . In this section we describe this complex in a way useful for calculations. Let  $\beta_a, \pi_a, \eta, \zeta$  be the canonical dual basis for  $\tilde{\mathfrak{h}}^*$ .

The Chevalley–Eilenberg differential is defined on generators as follows:

$$\begin{aligned} \partial R_{ab} &= \beta_a B_b - \beta_b B_a + \pi_a P_b - \pi_b P_a & \partial \zeta &= -\beta_a \pi_a \\ \partial B_a &= -\pi_a Z & \partial \beta_a &= 0 \\ \partial P_a &= \beta_a Z & \partial \pi_a &= 0 \\ \partial Z &= \partial H = 0 & \partial \eta &= 0. \end{aligned} \quad (58)$$

Let  $G \cong GL(\mathbb{R}^2) \ltimes \text{Aff}(\Lambda^2 \mathbb{R}^2)$  denote the subgroup of automorphisms of  $\tilde{\mathfrak{h}}$  which commutes with the action of  $\mathfrak{s}$ . It leaves  $R$  invariant and acts on  $\tilde{\mathfrak{h}}$  as follows:

$$(B, P, H, Z) \mapsto (B, P, H, Z) \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & \mu\Delta & \Delta \end{pmatrix}, \quad (59)$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(\mathbb{R}^2), \quad \Delta = ad - bc, \quad \mu \in \mathbb{R} \quad \text{and} \quad \lambda \in \mathbb{R}^\times. \quad (60)$$

The induced action on  $\tilde{\mathfrak{h}}^*$  is given by

$$(\beta, \pi, \eta, \zeta) \mapsto (\beta, \pi, \eta, \zeta) \begin{pmatrix} d/\Delta & -c/\Delta & 0 & 0 \\ -b/\Delta & a/\Delta & 0 & 0 \\ 0 & 0 & \lambda^{-1} & -\lambda^{-1}\mu \\ 0 & 0 & 0 & \Delta^{-1} \end{pmatrix}. \quad (61)$$

We proceed to enumerate the cochains.  $C^0$  is spanned by  $H, Z$ . The following tables enumerate the cochains in  $C^1, C^2$  and  $C^3$ . The primed cochains in Table 11 are only present for  $D = 5$  and as in the case of the static kinematical Lie algebra they will turn out not play any rôle in the calculations.

TABLE 9. Basis for  $C^1(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^{\mathfrak{s}}$

$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$
$\eta H$	$\eta Z$	$\zeta H$	$\zeta Z$	$\beta B$	$\beta P$	$\pi B$	$\pi P$

TABLE 10. Basis for  $C^2(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^{\mathfrak{s}}$

$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$	$c_9$	$c_{10}$	$c_{11}$	$c_{12}$	$c_{13}$	$c_{14}$	$c_{15}$
$\eta \zeta H$	$\eta \zeta Z$	$\eta \beta B$	$\eta \beta P$	$\eta \pi B$	$\eta \pi P$	$\zeta \beta B$	$\zeta \beta P$	$\zeta \pi B$	$\zeta \pi P$	$\beta \pi H$	$\beta \pi Z$	$\frac{1}{2} \beta \beta R$	$\beta \pi R$	$\frac{1}{2} \pi \pi R$

TABLE 11. Basis for  $C^3(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^{\mathfrak{s}}$

$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$	$b_7$	$b_8$	$b_9$	$b_{10}$	$b_{11}$
$\eta \zeta \beta B$	$\eta \zeta \beta P$	$\eta \zeta \pi B$	$\eta \zeta \pi P$	$\eta \beta \pi H$	$\eta \beta \pi Z$	$\eta \beta \beta R$	$\eta \beta \pi R$	$\eta \pi \pi R$	$\zeta \beta \pi H$	$\zeta \beta \pi Z$
$b_{12}$	$b_{13}$	$b_{14}$	$b_{15}$	$b_{16}$	$b_{17}$	$b_{18}$	$b'_{19}$	$b'_{20}$	$b'_{21}$	$b'_{22}$
$\zeta \beta \beta R$	$\zeta \beta \pi R$	$\zeta \pi \pi R$	$\beta \pi \beta B$	$\beta \pi \beta P$	$\beta \pi \pi B$	$\beta \pi \pi P$	$\epsilon \beta \beta \beta R$	$\epsilon \beta \beta \pi R$	$\epsilon \beta \pi \pi R$	$\epsilon \pi \pi \pi R$

$$\partial a_1 = \partial a_2 = \partial a_6 = \partial a_7 = 0 \quad \partial a_3 = -c_{11} \quad \partial a_4 = -c_{12} \quad \partial a_5 = c_{12} \quad \partial a_8 = c_{12}, \quad (62)$$
$$\begin{array}{lllll} \partial c_1 = b_5 & \partial c_4 = 0 & \partial c_7 = -b_{11} - b_{15} & \partial c_{10} = -b_{11} - b_{18} & \partial c_{13} = -b_{16} \\ \partial c_2 = b_6 & \partial c_5 = 0 & \partial c_8 = -b_{16} & \partial c_{11} = 0 & \partial c_{14} = b_{15} - b_{18} \\ \partial c_3 = -b_6 & \partial c_6 = -b_6 & \partial c_9 = -b_{17} & \partial c_{12} = 0 & \partial c_{15} = b_{17}, \end{array} \quad (63)$$
$$Z^2 = B^2 \oplus \mathcal{H}^2 \quad \text{where} \quad \mathcal{H}^2 = \mathbb{R} \langle c_2 + c_3, c_2 + c_6, c_4, c_5, c_7 - c_{10} + c_{14}, c_8 - c_{13}, c_9 + c_{15} \rangle. \quad (64)$$
$$\mathcal{H}^2 = \mathbb{R} \langle 2\mathbf{c}_2 + \mathbf{c}_3 + \mathbf{c}_6 \rangle \oplus \mathbb{R} \langle \mathbf{c}_7 - \mathbf{c}_{10} + \mathbf{c}_{14}, \mathbf{c}_8 - \mathbf{c}_{13}, \mathbf{c}_9 + \mathbf{c}_{15}, \mathbf{c}_3 - \mathbf{c}_6, \mathbf{c}_4, \mathbf{c}_5 \rangle, \quad (65)$$
$$(\mathbf{e}_1, \dots, \mathbf{e}_7) = (2\mathbf{c}_2 + \mathbf{c}_3 + \mathbf{c}_6, \mathbf{c}_7 - \mathbf{c}_{10} + \mathbf{c}_{14}, \mathbf{c}_8 - \mathbf{c}_{13}, \mathbf{c}_9 + \mathbf{c}_{15}, \mathbf{c}_3 - \mathbf{c}_6, \mathbf{c}_4, \mathbf{c}_5), \quad (66)$$
$$(\mathbf{e}_1, \dots, \mathbf{e}_7) = (\mathbf{e}_1, \dots, \mathbf{e}_7) \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & M_A & 0 \\ 0 & -\lambda^{-1} \mu \Delta M_A & \lambda^{-1} \Delta M_A \end{pmatrix}, \quad (67)$$
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad M_A = \frac{1}{\Delta^2} \begin{pmatrix} ad + bc & bd & -ac \\ 2cd & d^2 & -c^2 \\ -2ab & -b^2 & a^2 \end{pmatrix}. \quad (68)$$
$$\mathbf{K} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (69)$$

TABLE 12. Nijenhuis–Richardson •

[illegible]

**4.3. Infinitesimal deformations and obstructions.** We will parametrise the 7-dimensional space  $\mathcal{H}^2$  of infinitesimal deformations as

$$\varphi_1 = t_1(2c_2 + c_3 + c_6) + t_2(c_7 - c_{10} + c_{14}) + t_3(c_8 - c_{13}) + t_4(c_9 + c_{15}) + t_5(c_3 - c_6) + t_6c_4 + t_7c_5. \quad (70)$$

The first obstruction to integrability is  $\frac{1}{2}[\varphi_1, \varphi_1]$  whose vanishing (in cohomology, but in this case also on the nose) is equivalent to the following system of quadrics:

$$\begin{aligned} 0 &= 2t_1t_2 - t_3t_7 + t_4t_6 \\ 0 &= t_1t_3 + t_3t_5 - t_2t_6 \\ 0 &= t_1t_4 - t_4t_5 + t_2t_7. \end{aligned} \quad (71)$$

Since when these equations are satisfied  $[\varphi_1, \varphi_1] = 0$  (not just in cohomology) we can take  $\varphi_2 = 0$  and therefore there are no further obstructions to integrability.

To analyse this system of quadrics further we will exploit the action of the automorphism group  $G$ . It follows from the action of  $G$  on  $C^2$  that we can bring the triplet  $\mathbf{t} = (t_5, t_6, t_7)$  to one of four canonical forms, corresponding the causal type of  $\mathbf{t}$  relative to the lorentzian inner product defined by  $K$  in (69):

- (1) **Zero orbit:**  $\mathbf{t} = (0, 0, 0)$
- (2) **Spacelike orbit:**  $\mathbf{t} = (1, 0, 0)$
- (3) **Lightlike orbit:**  $\mathbf{t} = (0, 0, 1)$
- (4) **Timelike orbit:**  $\mathbf{t} = (0, 1, -1)$

**4.3.1. Zero orbit branch.** Here  $t_5 = t_6 = t_7 = 0$  and the system (71) of quadrics becomes  $t_1t_2 = t_1t_3 = t_1t_4 = 0$ , so we have two cases to consider depending on whether or not  $t_1 = 0$ .

If  $t_1 \neq 0$ , then  $t_2 = t_3 = t_4 = 0$  and by a judicious choice of  $\lambda \in \mathbb{R}^\times$  we can set  $t_1 = 1$  and the deformation is

$$\varphi = 2c_2 + c_3 + c_6 \quad (72)$$

which translates into the following additional brackets:

$$[H, B] = B \quad [H, P] = P \quad \text{and} \quad [H, Z] = 2Z, \quad (73)$$

which we recognise as a non-central extension of the kinematical Lie algebra (20) with  $\gamma = 1$ . Indeed,  $Z$  spans an ideal and quotienting by this ideal recovers the Lie algebra (20) with  $\gamma = 1$ .

If  $t_1 = 0$ , then we can use the automorphisms to bring  $\mathbf{t} = (t_2, t_3, t_4)$  to one of several canonical normal forms. First of all, notice that on the three-dimensional subspace of such  $\mathbf{t}$  the group  $G$  acts via  $\mathbf{t} \mapsto M_A \mathbf{t}$ , with  $M_A$  the matrix in (68). This defines an action of  $GL(\mathbb{R}^2)/\mathbb{R}^\times \cong SO_o(2, 1)$ , the identity component of the three-dimensional Lorentz group. Under the action of proper, orthochronous Lorentz transformations,  $\mathbb{R}^3$  breaks up into the following orbits:

- (1) **Zero orbit:**  $\mathbf{t} = (0, 0, 0)$ . This corresponds to no deformation at all.
- (2) **Spacelike orbits:**  $\mathbf{t} = (x, 0, 0)$  with  $x > 0$ . In this case, the deformation is given by

$$\varphi = x(c_7 - c_{10} + c_{14}) = x(\zeta\beta B - \zeta\pi P + \beta\pi R), \quad (74)$$

so that the brackets are

$$[Z, B] = xB \quad [Z, P] = -xP \quad \text{and} \quad [B, P] = Z + xR. \quad (75)$$

We can rescale  $Z \mapsto x^{-1}Z$  and  $P \mapsto x^{-1}P$  and in this way set  $x = 1$ . The resulting Lie brackets are

$$[Z, B] = B \quad [Z, P] = -P \quad \text{and} \quad [B, P] = Z + R, \quad (76)$$

which is isomorphic to a trivial central extension of  $\mathfrak{so}(D+1, 1)$  with central element  $H$ .

- (3) **Lightlike branches:**  $\mathbf{t} = (0, 0, \varepsilon)$ , where  $\varepsilon = \pm 1$ . The deformation cochain is

$$\varphi = \varepsilon(c_9 + c_{15}) = \varepsilon(\zeta\pi B + \frac{1}{2}\pi\pi R), \quad (77)$$

with Lie brackets

$$[Z, P] = \varepsilon B \quad [P, P] = \varepsilon R \quad \text{and} \quad [B, P] = Z. \quad (78)$$

We recognise these algebras as trivial central extensions of the euclidean (for  $\varepsilon = -1$ ) or Poincaré (for  $\varepsilon = +1$ ) algebras, with central element  $H$ .

- (4) **Timelike branches:**  $\mathbf{t} = (0, x, -x)$ , where  $x \in \mathbb{R}^\times$ . The deformation cochain is

$$\varphi = x(c_8 - c_9 - c_{13} - c_{15}) = x(\zeta\beta P - \zeta\pi B - \frac{1}{2}\beta\beta R - \frac{1}{2}\pi\pi R), \quad (79)$$

and Lie brackets

$$[Z, B] = xP \quad [Z, P] = -xB \quad [B, B] = -xR \quad \text{and} \quad [P, P] = -xR. \quad (80)$$

Let  $\varepsilon = -x/|x|$  be the sign of  $x$ . Rescaling  $B \mapsto (|x|)^{-1/2}B$ ,  $P \mapsto (|x|)^{-1/2}P$  and  $Z \mapsto x^{-1}Z$ , we may bring the brackets to one of two forms, depending on  $\varepsilon$ :

$$[Z, B] = P \quad [Z, P] = -B \quad [B, B] = \varepsilon R \quad [P, P] = \varepsilon R \quad \text{and} \quad [B, P] = \varepsilon Z. \quad (81)$$

We recognise these Lie algebras as trivial central extensions of  $\mathfrak{so}(D+2)$  (for  $\varepsilon = 1$ ) or  $\mathfrak{so}(D, 2)$  (for  $\varepsilon = -1$ ) with central element  $H$ .

4.3.2. *Spacelike orbit branches.* Here  $t_5 = 1$  and  $t_6 = t_7 = 0$  and the system (71) of quadrics becomes

$$(t_1 + 1)t_3 = 0 \quad (t_1 - 1)t_4 = 0 \quad \text{and} \quad t_1 t_2 = 0. \quad (82)$$

We therefore have several branches depending on the value of  $t_1$ .

(1) If  $t_1 \neq 0, \pm 1$ , then  $t_2 = t_3 = t_4 = 0$  and the deformation is

$$\varphi = 2t_1 c_2 + (t_1 + 1)c_3 + (t_1 - 1)c_6 = 2t_1 \eta \zeta Z + (t_1 + 1)\eta \beta B + (t_1 - 1)\eta \pi P, \quad (83)$$

with brackets

$$[H, B] = (t_1 + 1)B \quad [H, P] = (t_1 - 1)P \quad \text{and} \quad [H, Z] = 2t_1 Z. \quad (84)$$

We can bring this to a normal form by rescaling  $H$  and, if necessary, interchanging  $B$  and  $P$  and changing the sign of  $Z$ :

$$\boxed{[H, B] = \gamma B \quad [H, P] = P \quad [H, Z] = (1 + \gamma)Z \quad \text{and} \quad [B, P] = Z \quad \gamma \in (-1, 1).} \quad (85)$$

This Lie algebra is isomorphic to a non-central extension of the Lie algebra (20). Indeed, the quotienting by the ideal generated by  $Z$  gives the Lie algebra (20) with  $\gamma \in (-1, 1)$ .

(2) If  $t_1 = 0$ , then  $t_3 = t_4 = 0$ , so that the deformation is

$$\varphi = t_2(c_7 - c_{10} + c_{14}) + c_3 - c_6 = t_2(\zeta \beta B - \zeta \pi P + \beta \pi R) + \eta \beta B - \eta \pi P, \quad (86)$$

with brackets

$$[H, B] = B \quad [H, P] = -P \quad [Z, B] = t_2 B \quad [Z, P] = -t_2 P \quad [B, P] = Z + t_2 R. \quad (87)$$

Notice that  $Z - t_2 H$  is central. We must distinguish between two cases, depending on whether or not  $t_2 = 0$ :

- (a) if  $t_2 \neq 0$ , then we let  $Z \mapsto t_2^{-1} Z$  and  $H \mapsto H - t_2^{-1} Z$  and rescaling either  $B$  or  $P$  we can essentially set  $t_2 = 1$  and arrive at the Lie algebra given in (76).
- (b) if  $t_2 = 0$ , then we have

$$\boxed{[H, B] = B \quad [H, P] = -P \quad \text{and} \quad [B, P] = Z,} \quad (88)$$

which is isomorphic to a central extension of the Lie algebra (20) with  $\gamma = -1$ ; that is, to a central extension of the lorentzian Newton Lie algebra.

(3) If  $t_1 = 1$ , then  $t_2 = t_3 = 0$  and the deformation is

$$\varphi = 2c_2 + 2c_3 + t_4(c_9 + c_{15}) = 2\eta \zeta Z + 2\eta \beta B + t_4(\zeta \pi B + \frac{1}{2} \pi \pi R), \quad (89)$$

with brackets

$$[H, Z] = 2Z \quad [H, B] = 2B \quad [Z, P] = t_4 B \quad [P, P] = t_4 R. \quad (90)$$

We must distinguish between two cases, depending on whether or not  $t_4 = 0$ :

- (a) if  $t_4 = 0$ , then, rescaling  $H$ , we arrive at

$$[H, Z] = Z \quad [H, B] = B \quad \text{and} \quad [B, P] = Z, \quad (91)$$

which is isomorphic to (85) for  $\gamma = 0$ .

- (b) and if  $t_4 \neq 0$ , then introducing  $\varepsilon = t_4/|t_4|$ , we can bring the brackets to the following normal form:

$$\boxed{[H, Z] = Z \quad [H, B] = B \quad [Z, P] = \varepsilon B \quad [P, P] = \varepsilon R \quad \text{and} \quad [B, P] = Z.} \quad (92)$$

These Lie algebras are isomorphic to the extension of the euclidean or Poincaré Lie algebras by the dilatation  $H$ ; that is, they are isomorphic to  $\mathfrak{co}(D, 1) \ltimes \mathbb{R}^{D,1}$  or  $\mathfrak{co}(D + 1) \ltimes \mathbb{R}^{D+1}$ .

(4) If  $t_1 = -1$ , then  $t_2 = t_4 = 0$  and the deformation is

$$\varphi = -2c_2 - 2c_6 + t_3(c_8 - c_{13}) = -2\eta \zeta Z - 2\eta \pi P + t_3(\zeta \beta P - \frac{1}{2} \beta \beta R), \quad (93)$$

with brackets

$$[H, Z] = -2Z \quad [H, P] = -2P \quad [Z, B] = t_3 P \quad [B, B] = -t_3 R. \quad (94)$$

Exchanging  $B$  and  $P$  and changing the signs of  $H$  and  $Z$ , we see that this case leads to isomorphic algebras to the case  $t_1 = 1$ .

4.3.3. *Lightlike orbit branches.* Here  $t_5 = t_6 = 0$  and  $t_7 = 1$ . The system (71) of quadrics now sets  $t_2 = t_3 = 0$  and imposes  $t_1 t_4 = 0$ , which gives rise to two branches, depending on whether or not  $t_1 = 0$ :

(1)  $t_1 = 0$ : in this case the deformation is

$$\varphi = t_4(c_9 + c_{15}) + c_5 = t_4(\zeta\pi B + \frac{1}{2}\pi\pi R) + \eta\pi B, \quad (95)$$

so that the brackets are

$$[Z, P] = t_4 B \quad [H, P] = B \quad \text{and} \quad [P, P] = t_4 R. \quad (96)$$

We notice that  $Z - t_4 H$  is central, so this deformation will be a (possibly trivial) central extension. We must distinguish between two cases, according to whether or not  $t_4 = 0$ .

(a) If  $t_4 = 0$ , then we obtain

$$\boxed{[H, P] = B \quad \text{and} \quad [B, P] = Z,} \quad (97)$$

which is isomorphic to the **Bargmann** algebra: the universal central extension of the galilean algebra (22).

(b) If  $t_4 \neq 0$  and introducing  $\varepsilon = t_4/|t_4|$ , we may redefine generators to arrive at a Lie algebra isomorphic to (78).

(2)  $t_1 \neq 0$ : in this case  $t_4 = 0$  and the deformation is

$$\varphi = t_1(2c_2 + c_3 + c_6) + c_5 = t_1(2\eta\zeta Z + \eta\beta B + \eta\pi P) + \eta\pi B, \quad (98)$$

so that the brackets are

$$[H, Z] = 2t_1 Z \quad [H, B] = t_1 B \quad \text{and} \quad [H, P] = B + t_1 P. \quad (99)$$

We can actually absorb  $t_1$  into a redefinition of the generators and arrive at

$$\boxed{[H, B] = B \quad [H, P] = B + P \quad [H, Z] = 2Z \quad \text{and} \quad [B, P] = Z.} \quad (100)$$

This Lie algebra is isomorphic to a non-central extension of the Lie algebra (21), which we recover quotienting by the ideal generated by  $Z$ .

4.3.4. *Timelike orbit branches.* In this case  $t_5 = 0$ ,  $t_6 = 1$  and  $t_7 = -1$ . The system (71) of quadrics implies that  $t_2 = 0$ ,  $t_4 = -t_3$  and  $t_1 t_3 = 0$ . This then gives rise to two branches, depending on whether or not  $t_1 = 0$ :

(1)  $t_1 = 0$ : in this case the deformation is

$$\varphi = t_3(c_8 - c_9 - c_{13} - c_{15}) + c_4 - c_5 = t_3(\zeta\beta P - \zeta\pi B - \frac{1}{2}\beta\beta R - \frac{1}{2}\pi\pi R) + \eta\beta P - \eta\pi B, \quad (101)$$

with brackets

$$[Z, B] = t_3 P \quad [Z, P] = -t_3 B \quad [B, B] = -t_3 R \quad [P, P] = -t_3 R \quad [H, B] = P \quad [H, P] = -B. \quad (102)$$

We notice that  $Z - t_3 H$  is central for all  $t_3$ , so these Lie algebras will be (possibly trivial) central extensions. We distinguish between two cases depending on whether or not  $t_3 = 0$ .

(a) If  $t_3 = 0$ , then we obtain

$$\boxed{[H, B] = P \quad [H, P] = -B \quad \text{and} \quad [B, P] = Z.} \quad (103)$$

This Lie algebra can be interpreted as the central extension (with central element  $Z$ ) of the Lie algebra (23) with  $\alpha = 0$ ; that is, a central extension of the euclidean Newton Lie algebra.

(b) If  $t_3 \neq 0$ , then introducing  $\varepsilon = -t_3/|t_3|$ , we can rescale generators to arrive at a Lie algebra isomorphic to (81).

(2)  $t_1 \neq 0$ , so that  $t_3 = t_4 = 0$ , and the deformation is

$$\varphi = t_1(2c_2 + c_3 + c_6) + c_4 - c_5 = t_1(2\eta\zeta Z + \eta\beta B + \eta\pi P) + \eta\beta P - \eta\pi B \quad (104)$$

with brackets

$$[H, Z] = 2t_1 Z \quad [H, B] = t_1 B + P \quad [H, P] = t_1 P - B. \quad (105)$$

Here without loss of generality we can take  $t_1 = \alpha > 0$  and arrive at

$$\boxed{[H, B] = \alpha B + P \quad [H, P] = -B + \alpha P \quad [H, Z] = 2\alpha Z \quad \text{and} \quad [B, P] = Z.} \quad (106)$$

This is a non-central extension of the Lie algebra (23) (for  $\alpha > 0$ ) by the element  $Z$ .

**4.4. Invariant inner products.** We shall now investigate the existence of invariant inner products on the Lie algebras determined in this section, as we did in Section 2.6 for the kinematical Lie algebras classified in Section 2. We shall prove that only the trivial central extensions of the simple kinematical Lie algebras  $\mathfrak{so}(D+2)$ ,  $\mathfrak{so}(D+1,1)$  and  $\mathfrak{so}(D,2)$  admit invariant inner products. To prove that the other Lie algebras in Table 18 do not admit such inner products, we shall exploit the associativity condition (37). One of the immediate consequences of this condition is that for a Lie algebra  $\mathfrak{g}$  with an invariant inner product,  $\mathfrak{g}' = Z(\mathfrak{g})^\perp$ , where  $Z(\mathfrak{g})$  is the centre and  $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$  is the first derived ideal. Therefore if  $\mathfrak{g}$  is such that  $Z(\mathfrak{g}) = 0$  but  $\mathfrak{g}' \subsetneq \mathfrak{g}$ , then  $\mathfrak{g}$  cannot admit an invariant inner product. This is precisely the situation of the Lie algebras in the bottom third (below the line) of Table 18.

The first Lie algebra in the table (with brackets given by (57)) does not admit an invariant inner product. Indeed, if  $(-, -)$  is an associative symmetric bilinear form, it follows that

$$\delta_{ab}(Z, Z) = ([B_a, P_b], Z) = (B_a, [P_b, Z]) = 0 \quad (107)$$

and

$$\delta_{ab}(Z, H) = ([B_a, P_b], H) = (B_a, [P_b, H]) = 0, \quad (108)$$

so that  $(Z, -) = 0$ . The exact same calculation shows that in the Bargmann algebra (97) any associative symmetric bilinear form has  $(Z, -) = 0$ . A very similar argument shows that the trivial central extensions of the euclidean and Poincaré algebras (78) do not admit invariant inner products either. Indeed, if  $(-, -)$  is any associative symmetric bilinear form, then

$$\delta_{ab}(H, H) = ([B_a, P_b], H) = (B_a, [P_b, H]) = 0 \quad (109)$$

and

$$\delta_{ab}(H, Z) = ([B_a, P_b], Z) = (B_a, [P_b, Z]) = 0, \quad (110)$$

so that  $(H, -) = 0$ . The trivial central extensions of  $\mathfrak{so}(D+1,1)$ ,  $\mathfrak{so}(D+2)$  and  $\mathfrak{so}(D,2)$  do admit invariant inner products by taking the Killing form on the simple factor and some nonzero value for  $(Z, Z)$ .

Finally, we treat the centrally extended Newton algebras. The two cases are very similar, so we give details only for the case of the lorentzian algebra (88). Let  $(-, -)$  be an associative symmetric bilinear form. We will show that  $(B_a, -) = 0$ , so that it is degenerate. First of all, by rotational invariance,  $(B_a, H) = (B_a, Z) = 0$ . Let us calculate the others (in abbreviated notation)

$$\begin{aligned} (B, B) &= ([H, B], B) = (H, [B, B]) = 0 \\ (B, P) &= ([R, B], P) = (R, [B, P]) = (R, Z) = 0. \end{aligned} \quad (111)$$

The euclidean case (103) is similar. In summary, only the trivial central extensions of the simple kinematical Lie algebras  $\mathfrak{so}(D+1,1)$ ,  $\mathfrak{so}(D+2)$  and  $\mathfrak{so}(D,2)$  admit invariant inner products.

## 5. DEFORMATIONS OF THE CENTRALLY EXTENDED STATIC KINEMATICAL LIE ALGEBRA WITH $D = 4$

In this section we will let  $\tilde{\mathfrak{g}}$  denote the universal central extension of the static kinematical Lie algebra for  $D = 4$ . As in the non-centrally extended case,  $D = 4$  is slightly more complicated due to the semisimplicity of the rotation subalgebra  $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ . The notation is as in Section 3, in particular we shall let  $R_{ab}^\pm := \frac{1}{2} (R_{ab} \pm \frac{1}{2} \epsilon_{abcd} R_{cd})$  span  $\mathfrak{s} \cong \mathfrak{so}(4)$  and  $B_a, P_a, H, Z$  span the ideal  $\tilde{\mathfrak{h}}$  of  $\tilde{\mathfrak{g}}$ . As usual we will choose the canonical dual basis  $\beta_a, \pi_a, \eta, \zeta$  for  $\tilde{\mathfrak{h}}^*$ .

The nonzero Lie brackets in that basis are given by

$$\begin{aligned} [R_{ab}^\pm, R_{cd}^\pm] &= [R_{ab}, R_{cd}]^\pm \\ [R_{ab}^\pm, B_c] &= \frac{1}{2} (\delta_{bc} B_a - \delta_{ac} B_b \mp \epsilon_{abcd} B_d) \\ [R_{ab}^\pm, P_c] &= \frac{1}{2} (\delta_{bc} P_a - \delta_{ac} P_b \mp \epsilon_{abcd} P_d) \\ [B_a, P_b] &= \delta_{ab} Z. \end{aligned} \quad (112)$$

**5.1. The Chevalley–Eilenberg complex.** We apply the Hochschild–Serre decomposition theorem to calculate the infinitesimal deformations from the  $\mathfrak{s}$ -invariant subcomplex  $C^\bullet := C^\bullet(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^\mathfrak{s}$ .

We now proceed to enumerate bases for the spaces of cochains, noting that  $C^0$  is spanned by  $H$  and  $Z$ . The dimensions of  $C^1$ ,  $C^2$  and  $C^3$  are 8, 18 and 32, respectively, as can be checked using a roots and weights calculation. Natural bases are tabulated below.

TABLE 13. Basis for  $C^1(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^\mathfrak{s}$

$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$
$\eta H$	$\eta Z$	$\zeta H$	$\zeta Z$	$\beta B$	$\beta P$	$\pi B$	$\pi P$



TABLE 14. Basis for  $C^2(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^s$ 

$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$	$c_9$
$\eta\zeta H$	$\eta\zeta Z$	$\eta\beta B$	$\eta\beta P$	$\eta\pi B$	$\eta\pi P$	$\zeta\beta B$	$\zeta\beta P$	$\zeta\pi B$
$c_{10}$	$c_{11}$	$c_{12}$	$c_{13}$	$c_{14}$	$c_{15}$	$c_{16}$	$c_{17}$	$c_{18}$
$\zeta\pi P$	$\beta\pi H$	$\beta\pi Z$	$\frac{1}{2}\beta\beta R^+$	$\frac{1}{2}\beta\beta R^-$	$\beta\pi R^+$	$\beta\pi R^-$	$\frac{1}{2}\pi\pi R^+$	$\frac{1}{2}\pi\pi R^-$

TABLE 15. Basis for  $C^3(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^s$ 

$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$	$b_7$	$b_8$
$\eta\zeta\beta B$	$\eta\zeta\beta P$	$\eta\zeta\pi B$	$\eta\zeta\pi P$	$\eta\beta\pi H$	$\eta\beta\pi Z$	$\eta\beta\beta R^+$	$\eta\beta\beta R^-$
$b_9$	$b_{10}$	$b_{11}$	$b_{12}$	$b_{13}$	$b_{14}$	$b_{15}$	$b_{16}$
$\eta\beta\pi R^+$	$\eta\beta\pi R^-$	$\eta\pi\pi R^+$	$\eta\pi\pi R^-$	$\zeta\beta\pi H$	$\zeta\beta\pi Z$	$\zeta\beta\beta R^+$	$\zeta\beta\beta R^-$
$b_{17}$	$b_{18}$	$b_{19}$	$b_{20}$	$b_{21}$	$b_{22}$	$b_{23}$	$b_{24}$
$\zeta\beta\pi R^+$	$\zeta\beta\pi R^-$	$\zeta\pi\pi R^+$	$\zeta\pi\pi R^-$	$\beta\pi\beta B$	$\beta\pi\beta P$	$\beta\pi\pi B$	$\beta\pi\pi P$
$b_{25}$	$b_{26}$	$b_{27}$	$b_{28}$	$b_{29}$	$b_{30}$	$b_{31}$	$b_{32}$
$\epsilon\beta\beta\beta R^+$	$\epsilon\beta\beta\beta R^-$	$\epsilon\beta\beta\pi R^+$	$\epsilon\beta\beta\pi R^-$	$\epsilon\beta\pi\pi R^+$	$\epsilon\beta\pi\pi R^-$	$\epsilon\pi\pi\pi R^+$	$\epsilon\pi\pi\pi R^-$

The Chevalley–Eilenberg differential  $\partial : C^p \rightarrow C^{p+1}$  is defined on generators by

$$\partial\eta = \partial\beta_a = \partial\pi_a = \partial H = \partial Z = 0 \quad (113)$$

and

$$\begin{aligned} \partial R_{ab}^\pm &= \frac{1}{2}(\beta_a B_b - \beta_b B_a + \pi_a P_b - \pi_b P_a \pm \epsilon_{abcd}(\beta_c B_d + \pi_c P_d)) \\ \partial B_a &= -\pi_a Z \\ \partial P_a &= \beta_a Z \\ \partial \zeta &= -\beta_a \pi_a. \end{aligned} \quad (114)$$

The Chevalley–Eilenberg differential  $\partial : C^1 \rightarrow C^2$  is given on the basis by

$$\partial a_1 = \partial a_2 = \partial a_6 = \partial a_7 = 0 \quad \partial a_5 = \partial a_8 = c_{12} \quad \partial a_4 = -c_{12} \quad \text{and} \quad \partial a_3 = -c_{11}, \quad (115)$$

from where we see that  $B^2 = \mathbb{R}\langle c_{11}, c_{12} \rangle$ . The differential  $\partial : C^2 \rightarrow C^3$  is given on the basis by

$$\begin{aligned} \partial c_1 &= b_5 & \partial c_7 &= -b_{14} - b_{21} & \partial c_{13} &= -\frac{1}{2}b_{22} + \frac{1}{4}(b_{25} + b_{28}) \\ \partial c_2 &= b_6 & \partial c_8 &= -b_{22} & \partial c_{14} &= -\frac{1}{2}b_{22} - \frac{1}{4}(b_{25} + b_{28}) \\ \partial c_3 &= -b_6 & \partial c_9 &= -b_{23} & \partial c_{15} &= \frac{1}{2}(b_{21} - b_{24} + b_{27} + b_{30}) \\ \partial c_4 &= 0 & \partial c_{10} &= -b_{14} - b_{24} & \partial c_{16} &= \frac{1}{2}(b_{21} - b_{24} - b_{27} - b_{30}) \\ \partial c_5 &= 0 & \partial c_{11} &= 0 & \partial c_{17} &= -\frac{1}{2}b_{23} + \frac{1}{4}(b_{29} + b_{32}) \\ \partial c_6 &= -b_6 & \partial c_{12} &= 0 & \partial c_{18} &= -\frac{1}{2}b_{23} - \frac{1}{4}(b_{29} + b_{32}) \end{aligned} \quad (116)$$

The last piece of data that we need is the Nijenhuis–Richardson bracket  $\llbracket -, - \rrbracket : C^2 \times C^2 \rightarrow C^3$ . Table 16 collects the calculations of  $c_i \bullet c_j$ , from where we can read off  $\llbracket c_i, c_j \rrbracket$  by symmetrisation.

**5.2. Infinitesimal deformations.** From the action of the Chevalley–Eilenberg differential  $\partial$  on  $C^2$ , described in equation (116), we find that

$$\begin{aligned} Z^2 &= B^2 \oplus \mathbb{R}\langle 2c_2 + c_3 + c_6 \rangle \oplus \mathbb{R}\langle c_3 - c_6, c_4, c_5 \rangle \\ &\quad \oplus \langle c_7 - c_{10} + c_{15} + c_{16}, c_8 - c_{13} - c_{14}, c_9 + c_{17} + c_{18} \rangle. \end{aligned} \quad (117)$$

Therefore the most general infinitesimal deformation can be parametrised as

$$\begin{aligned} \varphi_1 &= t_1(2c_2 + c_3 + c_6) + t_2(c_7 - c_{10} + c_{15} + c_{16}) \\ &\quad + t_3(c_8 - c_{13} - c_{14}) + t_4(c_9 + c_{17} + c_{18}) + t_5(c_3 - c_6) + t_6 c_4 + t_7 c_5. \end{aligned} \quad (118)$$

TABLE 16. Nijenhuis–Richardson •

•	c <sub>1</sub>	c <sub>2</sub>	c <sub>3</sub>	c <sub>4</sub>	c <sub>5</sub>	c <sub>6</sub>	c <sub>7</sub>	c <sub>8</sub>	c <sub>9</sub>	c <sub>10</sub>	c <sub>11</sub>	c <sub>12</sub>	c <sub>13</sub>	c <sub>14</sub>	c <sub>15</sub>	c <sub>16</sub>	c <sub>17</sub>	c <sub>18</sub>
c <sub>1</sub>	0	0	b <sub>1</sub>	b <sub>2</sub>	b <sub>3</sub>	b <sub>4</sub>	0	0	0	0	0	0	0	0	0	0	0	0
c <sub>2</sub>	0	0	0	0	0	0	b <sub>1</sub>	b <sub>2</sub>	b <sub>3</sub>	b <sub>4</sub>	0	0	0	0	0	0	0	0
c <sub>3</sub>	0	0	0	0	0	0	b <sub>1</sub>	b <sub>2</sub>	0	0	b <sub>5</sub>	b <sub>6</sub>	b <sub>7</sub>	b <sub>8</sub>	b <sub>9</sub>	b <sub>10</sub>	0	0
c <sub>4</sub>	0	0	0	0	0	0	0	0	b <sub>1</sub>	b <sub>2</sub>	0	0	0	0	b <sub>7</sub>	b <sub>8</sub>	b <sub>9</sub>	b <sub>10</sub>
c <sub>5</sub>	0	0	0	0	0	0	b <sub>3</sub>	b <sub>4</sub>	0	0	0	0	b <sub>9</sub>	b <sub>10</sub>	b <sub>11</sub>	b <sub>12</sub>	0	0
c <sub>6</sub>	0	0	0	0	0	0	0	0	b <sub>3</sub>	b <sub>4</sub>	b <sub>5</sub>	b <sub>6</sub>	0	0	b <sub>9</sub>	b <sub>10</sub>	b <sub>11</sub>	b <sub>12</sub>
c <sub>7</sub>	0	0	−b <sub>1</sub>	−b <sub>2</sub>	0	0	0	0	0	0	b <sub>5</sub>	b <sub>6</sub>	b <sub>15</sub>	b <sub>16</sub>	b <sub>17</sub>	b <sub>18</sub>	0	0
c <sub>8</sub>	0	0	0	0	−b <sub>1</sub>	−b <sub>2</sub>	0	0	0	0	0	0	0	0	b <sub>15</sub>	b <sub>16</sub>	b <sub>17</sub>	b <sub>18</sub>
c <sub>9</sub>	0	0	−b <sub>3</sub>	−b <sub>4</sub>	0	0	0	0	0	0	0	0	b <sub>17</sub>	b <sub>18</sub>	b <sub>19</sub>	b <sub>20</sub>	0	0
c <sub>10</sub>	0	0	0	0	−b <sub>3</sub>	−b <sub>4</sub>	0	0	0	0	b <sub>13</sub>	b <sub>14</sub>	0	0	b <sub>17</sub>	b <sub>18</sub>	b <sub>19</sub>	b <sub>20</sub>
c <sub>11</sub>	b <sub>13</sub>	b <sub>14</sub>	b <sub>21</sub>	b <sub>22</sub>	b <sub>23</sub>	b <sub>24</sub>	0	0	0	0	0	0	0	0	0	0	0	0
c <sub>12</sub>	−b <sub>5</sub>	−b <sub>6</sub>	0	0	0	0	b <sub>21</sub>	b <sub>22</sub>	b <sub>23</sub>	b <sub>24</sub>	0	0	0	0	0	0	0	0
c <sub>13</sub>	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
c <sub>14</sub>	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
c <sub>15</sub>	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
c <sub>16</sub>	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
c <sub>17</sub>	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
c <sub>18</sub>	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

5.3. **Obstructions.** The first obstruction is the class in  $H^3$  of  $\frac{1}{2}[\varphi_1, \varphi_1] = \varphi_1 \bullet \varphi_1$ , which we can calculate from the explicit expression 16 for the Nijenhuis–Richardson • product. Doing so, we find that  $\varphi_1 \bullet \varphi_1$  vanishes in cohomology if and only if it vanishes on the nose and this happens if and only if the following quadric equations hold:

$$\begin{aligned}
0 &= 2t_1t_2 - t_3t_7 + t_4t_6 \\
0 &= t_1t_3 + t_3t_5 - t_2t_6 \\
0 &= t_1t_4 - t_4t_5 + t_2t_7.
\end{aligned} \tag{119}$$

If they are satisfied and since  $[\varphi_1, \varphi_1] = 0$ , we can take  $\varphi_2 = 0$  and the deformation integrates already to first order.

We now observe that since  $R^+ + R^- = R$ , and the sums of cochains appearing in  $\varphi_1$  are

$$c_{13} + c_{14} = \frac{1}{2}\beta\pi R \quad c_{15} + c_{16} = \beta\pi R \quad \text{and} \quad c_{17} + c_{18} = \frac{1}{2}\pi\pi R, \tag{120}$$

the expression (118) above for  $\varphi_1$  coincides *mutatis mutandis* with the one in equation (70). Furthermore the conditions (119) and (71) are identical, so that the rest of the analysis proceeds as in the case  $D \geq 5$ . As in that case, here too the only Lie algebras admitting an invariant inner product are the trivial central extensions of the simple kinematical Lie algebras.

## 6. SUMMARY AND CONCLUSIONS

Deformation theory provides a powerful and systematic approach to classifying Lie algebras. When the Lie algebras in question have a “sizeable” semisimple subalgebra, the calculations are particularly tractable due to the Hochschild–Serre spectral sequence, which guarantees that we can work with a quasi-isomorphic subcomplex of the deformation complex which is typically much smaller.

In this paper we have applied these techniques to classify kinematical Lie algebras in dimension  $D \geq 4$  (up to isomorphism) by classifying deformations of the static kinematical Lie algebra  $\mathfrak{g}$ . The Lie algebra  $\mathfrak{g}$  admits a universal central extension  $\tilde{\mathfrak{g}}$  and we have classified its deformations as well. This gives rise to a number of extensions (trivial, central and non-central) of deformations of  $\mathfrak{g}$ .

Let us summarise the results obtained in this paper. First of all we summarise the kinematical Lie algebras. It is convenient to tabulate them to ease comparison with the classical  $D = 3$  results and also with the results for  $D = 2$  [4]. Table 17 lists the isomorphism classes of kinematical Lie algebras. In some cases we have changed basis to bring the Lie algebra to a more familiar form. Comparing with Table 1 in [3] or Table 1 in [1], we see that, unsurprisingly, there are some kinematical Lie algebras in  $D = 3$  which do not exist in  $D \geq 4$ . Those additional  $D = 3$  Lie algebras owe their existence to the vector product  $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , which is absent in  $D > 3$ . (There is a vector product in  $\mathbb{R}^7$ , but it is not invariant under  $\mathfrak{so}(7)$  but only under a  $\mathfrak{g}_2$  subalgebra.) Comparing with

Table 1 in [4], we see that also in  $D = 2$  there are additional kinematical Lie algebras which owe their existence this time to the symplectic structure on  $\mathbb{R}^2$ . (There is a symplectic structure on  $\mathbb{R}^D$  for any even  $D$ , but only for  $D = 2$  it is  $\mathfrak{so}(D)$ -invariant.)

TABLE 17. Kinematical Lie algebras in  $D \geq 4$ 

Eq.	Nonzero Lie brackets					Comments	Metric?
5						static	
22	$[H, B] = P$					galilean	
20	$[H, B] = \gamma B$	$[H, P] = P$				$\gamma \in (-1, 1]$	
20	$[H, B] = -B$	$[H, P] = P$				lorentzian Newton	
23	$[H, B] = B + P$	$[H, P] = P$					
21	$[H, B] = \alpha B + P$	$[H, P] = \alpha P - B$				$\alpha > 0$	
21	$[H, B] = P$	$[H, P] = -B$				euclidean Newton	
32	$[B, P] = H$					Carroll	
36		$[H, P] = B$	$[B, P] = H$	$[P, P] = R$	$\mathfrak{e}$		
36		$[H, P] = -B$	$[B, P] = H$	$[P, P] = -R$	$\mathfrak{p}$		
34	$[H, B] = B$	$[H, P] = -P$	$[B, P] = H + R$			$\mathfrak{so}(D + 1, 1)$	✓
35	$[H, B] = -P$	$[H, P] = B$	$[B, P] = H$	$[B, B] = R$	$[P, P] = R$	$\mathfrak{so}(D + 2)$	✓
35	$[H, B] = P$	$[H, P] = -B$	$[B, P] = H$	$[B, B] = -R$	$[P, P] = -R$	$\mathfrak{so}(D, 2)$	✓

As shown in Section 2.6, only the simple Lie algebras in this list (i.e.,  $\mathfrak{so}(D+1, 1)$ ,  $\mathfrak{so}(D+2)$  and  $\mathfrak{so}(D, 2)$ ) admit an associative (i.e., ad-invariant) inner product. This is in sharp contrast with  $D \leq 3$ , where there are a number of nonsimple metric kinematical Lie algebras.

Next we summarise the deformations of the central extension of the static kinematical Lie algebra with  $D \geq 4$ . Table 18 lists the isomorphism classes of these deformations with an identifying comment as to their structure or their name, when known. All of these Lie algebras share the following Lie brackets (in abbreviated notation):

$$[R, R] = R \quad [R, B] = B \quad [R, P] = P \quad [R, H] = 0 \quad \text{and} \quad [R, Z] = 0. \quad (121)$$

In the table we will only list any additional nonzero brackets. In some cases we have changed notations ( $H$  for  $Z$  and  $B$  for  $P$ ) for the sake of uniformity. The table is divided into three: the top third consists of (nontrivial) central extensions, the middle third of trivial central extensions and the bottom third of non-central extensions of kinematical Lie algebras.

Comparing with Table 2 in [1], we see that contrary to the deformations of the static kinematical Lie algebra (without central extension), the results in  $D = 3$  are *mutatis mutandis* the same as  $D \geq 4$ . The similar classification in  $D = 2$  does not exist: the universal central extension of the  $D = 2$  static kinematical Lie algebra has five central generators and not just one.

TABLE 18. Deformations of the centrally extended static kinematical Lie algebra in  $D \geq 4$ 

Eq.	Nonzero Lie brackets					Comments	Metric?
57	$[\mathbf{B}, \mathbf{P}] = \mathbf{Z}$					centrally extended static	
88	$[\mathbf{B}, \mathbf{P}] = \mathbf{Z}$	$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{P}$			central extension of lorentzian Newton	
103	$[\mathbf{B}, \mathbf{P}] = \mathbf{Z}$	$[\mathbf{H}, \mathbf{B}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{B}$			central extension of euclidean Newton	
97	$[\mathbf{B}, \mathbf{P}] = \mathbf{Z}$	$[\mathbf{H}, \mathbf{B}] = -\mathbf{P}$			Bargmann		
78	$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$	$[\mathbf{H}, \mathbf{B}] = \mathbf{P}$	$[\mathbf{B}, \mathbf{B}] = \mathbf{R}$		$\mathfrak{p} \oplus \mathbb{R}$		
78	$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$	$[\mathbf{H}, \mathbf{B}] = -\mathbf{P}$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{R}$		$\mathfrak{e} \oplus \mathbb{R}$		
76	$[\mathbf{B}, \mathbf{P}] = \mathbf{H} + \mathbf{R}$	$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{P}$			$\mathfrak{so}(D+1, 1) \oplus \mathbb{R}$	✓
81	$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$	$[\mathbf{H}, \mathbf{B}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{B}$	$[\mathbf{B}, \mathbf{B}] = \mathbf{R}$	$[\mathbf{P}, \mathbf{P}] = \mathbf{R}$	$\mathfrak{so}(D+2) \oplus \mathbb{R}$	✓
81	$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$	$[\mathbf{H}, \mathbf{B}] = -\mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{B}$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{R}$	$[\mathbf{P}, \mathbf{P}] = -\mathbf{R}$	$\mathfrak{so}(D, 2) \oplus \mathbb{R}$	✓
73	$[\mathbf{B}, \mathbf{P}] = \mathbf{Z}$	$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{Z}] = 2\mathbf{Z}$			
85	$[\mathbf{B}, \mathbf{P}] = \mathbf{Z}$	$[\mathbf{H}, \mathbf{B}] = \gamma \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{Z}] = (\gamma + 1)\mathbf{Z}$	$\gamma \in (-1, 1)$		
100	$[\mathbf{B}, \mathbf{P}] = \mathbf{Z}$	$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{B} + \mathbf{P}$	$[\mathbf{H}, \mathbf{Z}] = 2\mathbf{Z}$			
106	$[\mathbf{B}, \mathbf{P}] = \mathbf{Z}$	$[\mathbf{H}, \mathbf{B}] = \alpha \mathbf{B} + \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{B} + \alpha \mathbf{P}$	$[\mathbf{H}, \mathbf{Z}] = 2\alpha \mathbf{Z}$	$\alpha > 0$		
92	$[\mathbf{B}, \mathbf{P}] = \mathbf{Z}$	$[\mathbf{Z}, \mathbf{B}] = -\mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{Z}] = \mathbf{Z}$	$[\mathbf{B}, \mathbf{B}] = \mathbf{R}$	$\mathfrak{co}(D, 1) \ltimes \mathbb{R}^{D, 1}$	
92	$[\mathbf{B}, \mathbf{P}] = \mathbf{Z}$	$[\mathbf{Z}, \mathbf{B}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{Z}] = \mathbf{Z}$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{R}$	$\mathfrak{co}(D+1) \ltimes \mathbb{R}^{D+1}$	

As in the case of kinematical Lie algebras, the only Lie algebras in this table which admit an invariant inner product are the trivial central extensions of the simple kinematical Lie algebras. This agrees with the results for  $D = 3$  as well. We suspect that the case of  $D = 2$  will provide us with some nonsimple metric Lie algebras, but we have not classified those deformations yet.

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